Technical Notes

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Predicting the Optimum Step Size for the Numerical Solution of Steady-State Fluid Flow Problems

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Introduction

IN the numerical solution of the nonlinear time-dependent Navier-Stokes equations for viscous incompressible fluids, methods involving implicit time-step formulations are generally regarded as preferable, since the more satisfactory stability properties allow much larger time-step sizes to be used.[1] However, it is not true that large time increments are invariably better, since the subsidiary calculations necessary at each step may be excessive. An explicit formulation, limited to a small step size but involving far fewer computations at each step, may actually be less expensive of computer time. This can be the case, of course, only if the largest time step consistent with numerical stability is used.

A precise analysis leading to a prediction of this maximum time step is not available, because of the nonlinearity of the system. However, empirical[2] and theoretical[3] arguments exist which show how this optimal step size can be calculated by searching through the current values of the velocity components. An explicit procedure using such a time step turns out to be more than competitive with standard implicit procedures.

Formulation

We consider the problem of two-dimensional steady flow in a square cavity, driven by a velocity on the upper surface. In the stream function-vorticity formulation the dimensionless governing equations are[4]

$$\partial\omega/\partial t = \partial(\psi,\omega)/\partial(x,y) + (1/R)\nabla^2\omega \quad (1)$$

$$\nabla^2\psi = \omega \quad (2)$$

with boundary conditions which have been treated elsewhere.[4]

For steady-state problems Eq. (1) might be replaced by

$$\nabla^2\omega = -R\,\partial(\psi,\omega)/\partial(x,y) \quad (3)$$

and solved together with Eq. (2) by standard methods for elliptic equations. This approach is reasonable at small Reynolds numbers, but difficulties arise[5] for large values of R. Then, it is better[6] to solve the full initial-value problem in Eqs. (1) and (2), even if only the steady-state solution is of interest. In this case Eq. (2) is solved by some relaxation procedure involving several approximations, Eq. (1) is used to calculate a new vorticity field and the procedure is repeated. It should be mentioned that high accuracy in these intermediate solutions of Eq. (2) is not justified.

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Unlike their implicit counterparts, explicit methods require a fixed amount of computer time per time step to produce a new vorticity field from Eq. (1); additional computing time at each step is required only because the larger changes of ω require more iterations of Eq. (2). However, it turns out[3,5] that one can regulate this by iterating Eq. (2) a fixed number of times, say once or twice, without harming the convergence of the method. Thus, greatest computer efficiency is obtained when Eq. (1) is on the verge of instability; this is not true with other methods.

Although the total system is nonlinear, Eq. (1) is linear as far as the selection of Δt is concerned, for ω is temporarily known. Thus the analysis referred to above[2,3] may be applied; we must have

$$\Delta t/Rh^2 < \min\,[1/4,\ 4/R^2h^2(u_{ij} + v_{ij})^2] \quad (4)$$

for stability. Here the values of i and j which must be used are those which produce the smallest value for the second entry in (4).

Results

Strictly experimental runs were made for fixed time increments to determine speed of convergence for several Reynolds numbers. In addition, values for Δt were produced from Eq. (4) using a search technique which omitted the top row of the mesh, on which the velocity is specified. The values for Δt resulting from this search are labelled "S" in Fig. 1 for $R = 100$. In this figure, we plot the logarithm (base ten) of the total residual against the step size. The total residual is defined as the sum over all interior mesh points of the residual of the fourth-order Eq. (3), the finite-difference form of

$$R_{ij} = \{\nabla^4\psi + R\,\partial(\psi,\nabla^2\psi)/\partial(x,y)\}_{ij} \quad (5)$$

The logarithm provides a more presentable display, since the error tends to decrease by a constant factor in a fixed number of iterations. It is found in each case that the "search" results provide highly accurate estimates of the maximum permissible value.‡

As expected, the values produced for Δt change as the solution progresses, but is interesting to note that the change is not large. Although the system is strongly nonlinear, the effect of the nonlinearity on stability is mild. Another way of saying this is that the most critical region for stability (the high-velocity corner) is close to the source of the motion and quick to reach something like its ultimate state.

The economy held up in more realistic cases for which the experimentation of Fig. 1 would have been prohibitively expensive. Runs on a grid of 40 × 40 points, for $R = 1000$, converged (average residual per mesh point $<10^{-5}$) after 1800 time steps, a total time of less than two hours on an IBM 7040. This is better (by a factor of more than two) than is normally achieved using implicit methods, and compares favorably to published results.[6] Explicit time-step formulations, long dismissed as inefficient, may be worthy of renewed consideration.

‡ This Reynolds number is large enough to make nonlinear effects dominate, as is shown by the fact that the linearized analysis, leading to the first term in Eq. (4), gives $\Delta t = 0.25$ as the stability limit. The computation is completely unstable for this step size.

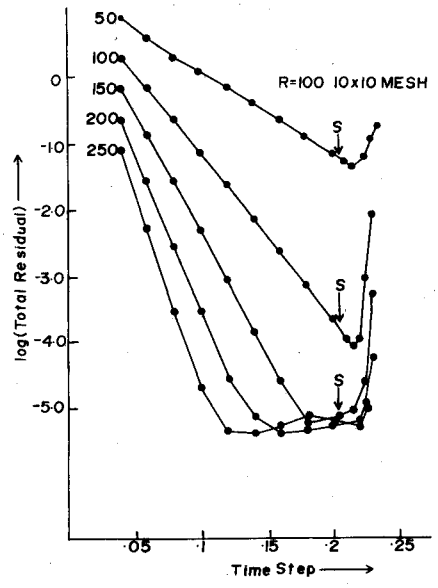

Fig. 1 Effect of step size on convergence. Each curve represents a fixed number of time steps and a constant amount of computer time. S = search prediction. Logarithms are to base ten.

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Similar Solutions in Vibrational Nonequilibrium Nozzle Flows

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Nomenclature

A = area ratio
C,D = constants, for a given gas
L = nozzle scale parameter ($L = r_*'/\tan\gamma$)
M = Mach number
m = molecular weight
p = pressure
R = gas constant
S_0 = reservoir entropy ($S_0 = S_0'/R$)
T_t = translational temperature
T_v = vibrational temperature
u = velocity ($u = u'/u_0'$)
u_0' = velocity defined as $(p_0'/\rho_0')^{1/2}$
x = distance along the nozzle axis ($x = x'/L$)
ρ = density
θ_v = characteristic vibrational temperature ($\theta_v = h\nu/k$)
ϕ = vibrational temperature function ($\phi = \theta_v/T_v'$)
ψ = translational temperature function ($\psi = \theta_v/T_t'$)

Subscripts

* = nozzle throat
0 = reservoir

Superscript

( )′ = dimensional quantity

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Introduction

In the past decade a considerable amount of theoretical effort has been directed toward understanding the nonequilibrium flow effects in nozzles under steady flow conditions. The problem has been studied under the assumption of pseudo-one-dimensional, adiabatic, inviscid flow. In spite of these simplifying assumptions the solutions are far from being simple and are often plagued by many numerical procedural difficulties.[1–3] A comprehensive review of this problem is presented in Ref 4. More recently, a time-dependent analysis[5] has been proposed which circumvents some of the numerical difficulties but retains the problem of determining the flow quantities through a numerous-stepped process. The present state-of-the-art for solving vibrational nonequilibrium nozzle flows requires complex computer programs with which the flow variables are determined by numerical integration for any given initial and boundary conditions. However, this approach does not provide suitable theoretical comparisons for use by the experimentalist because of the many variables involved. Thus, it is apparent that general correlating parameters are needed.

In the present analysis, the governing equations, for a pseudo-one-dimensional nonequilibrium nozzle flow with vibrational energy relaxation but no dissociation, are transformed into a similar form by using a new independent variable η. It can be shown that the similar solutions, for a family of nozzle shapes and a specified gas, depend on two parameters, s_0 and λ, in addition to the independent variable η. However, the similar equations are further reduced to a near universal form by a transformation of the independent variable η to ξ so that the similar solutions depend on a single parameter χ with ξ as the independent variable. General similar solutions which can be used for all combinations of initial conditions are presented in a single graph for nitrogen.

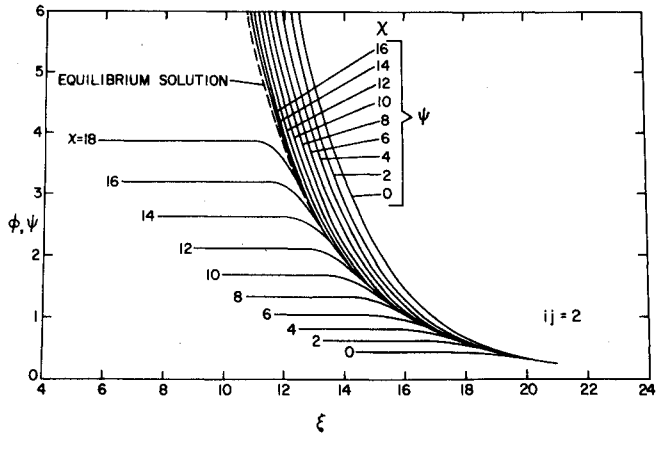

Fig. 1 Similar solutions for vibrational nonequilibrium nitrogen flow (D = 14.7).